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COMMENT

Upper critical dimension of Kauffman cellular automata

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Abstract. Random Boolean networks on nearest-neighbour d-dimensional lattices are argued to belong to the universality class of directed percolation with quenched disorder in $d+1$ dimensions. Hansen's computer simulations for $d = 4$ are thus interpreted as being at the upper critical dimension.

In cellular automata, each lattice site carries a 'spin' which may point either up or down. The orientation of a spin at time $t+1$ is determined completely by the orientation of its neighbour spins at time t. On a lattice with K such neighbours one has $N = 2^K$ neighbour configurations and thus 2^N different rules which the spins might obey. In the Kauffman model [1], each site selects randomly at the beginning which of these 2^N rules it wants to obey. One may bias this random selection by assuming that each site, for each of the N neighbour configurations separately, selects with probability p the rule giving an up spin, and with probability $1-p$ the rule selecting a down spin. In the 'quenched' Kauffman model this selection is made, once and for all, at the beginning, whereas in the 'annealed' Kauffman model this selection of rules is made again and again at every time step [2]. Since then simply at each time step the spin points up with probability p and down with probability $1-p$, the annealed Kauffman model on a d-dimensional lattice is related to directed percolation [3] in $d+1$ dimensions, the time being the additional dimension.

This relation arises if one investigates the Hamming distance between two different configurations studied on the same lattice with the same set of rules [2]. This distance is the number of spins differing in a site-by-site comparison of the two configurations. If initially the two configurations differ by a single spin only, one may call this Hamming distance the 'damage' spreading through the system as a consequence of this single spin flip. For the annealed Kauffman model the damage spreads along the clusters of a directed percolation lattice [4], where bonds are present randomly with probability $2p(1-p)$. We want to postulate a similar relation for the quenched Kauffman model.

We start with a completely random spin distribution: spin up at site i with probability $\frac{1}{2}$. In the quenched Kauffman model each site i gives, at the next time step, an up spin with a probability p_i depending on the particular rule it has selected for itself. If the neighbour configuration would remain random then the spin of that given site would point up with this same time-independent probability. Although the local configurations are not really random one can still define a probability for a spin to be up by averaging over all the possible initial configurations; and although this value will not be completely time independent on the time average there will be sites i that have

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a probability p_i distinctly above or below average. In this sense one can define p_i such that different sites i and k have different but time-independent probabilities p_i and p_k for pointing up. If we average over all sites i, the average probability is $\langle p_i \rangle = m$, but for each site it can be different. For example, one particular site could have selected the rule 'always up' and thus has $p_i = 1$; it actually has selected that extreme rule with probability p^N.

We now simulate the spreading of damage by looking at two lattices simultaneously, employing the same rules for two lattices that initially differ only on very few sites in their spin orientation; then the damage spreads from one spin to a neighbour with some finite probability, if the two neighbourhoods are not identical. We call this probability p_i' for site i (meant in the same sense as above) and its average over all sites the damage-spreading probability p'. If the neighbourhood configurations would be random, p_i' would be $2p_i(1 - p_i)$ and would be fixed in time, depending only on the rule selected by that site initially. The time can be interpreted as an additional spatial dimension. This situation thus corresponds to quenched disordered directed percolation in $d + 1$ dimensions [5, 6] with probability p'.

If the damage reaches a site for the first time, the *a priori* probability for this damage to be transferred is p'. If the damage reaches a point twice, then with a probability that we call $1/N'$ it will be subjected to the same behaviour as before. We call N' the effective number of possible neighbour configurations. One could estimate N' by summing over many (M) initial configurations and measure for each site how often the various possible neighbour configurations actually occur. Those appearing at least M/N times are counted as effective. N' can be much less than the total number of neighbour configurations, $N = 2^K$, and depends on p; for example $N' = 1$ at $p = 0$. This dependence could cause a phase transition as a function of $1/N'$. N' is also diminished by the fact that the neighbour configurations are correlated and not random. Thus $1/N'$ can be considered as a measure for the quenched disorder. In terms of the calculations of [6] it corresponds to the square of s_0, where s_0 is the average dispersion of quenched percolation probabilities at different sites. We must, however, be very careful in applying this approach to systems with large disorder. Usually, disorder manifests itself as an additional effective attraction in the theory. The renormalisation group calculation shows [6] that the fixed point which determines the universality class of disordered directed percolation is very close to being unstable. So it may happen that at a certain degree of disorder there is no stable solution at all. For deterministic cellular automata this effect will lead to short limit cycles of the system. Essentially, strong disorder from this point of view is enhanced in low-dimensional lattices with low coordination numbers K; indeed in the honeycomb lattice, $K = 3$, no chaotic phase with damage spreading over the whole lattice was found [8].

In our case the behaviour of neighbours is correlated. Small limit cycles are possible for certain regions or for the whole lattice. Nevertheless the total concentration of up spins in the Kauffman model is known to be close to the probability p that a rule gives the value spin up. The correlations for the neighbour configurations can, at least partly, be taken into account by our distinction between N and N'. We *conjecture* that this difference takes into account the main effect of these correlations; the larger the number of neighbours is, the less important we expect the correlations to be.

Similarly, for isotropic percolation one may look at correlated instead of random percolation, where (as in the Ising model or lattice gas) spins tend to be parallel, i.e. occupied sites attract each other. A correlated percolation model has different thresholds than random percolation but is believed to belong to the same universality class

(same fractal dimensions, etc) as random percolation as long as no phase separation sets in. Similarly we expect damage spreading in the quenched Kauffman model not to be identical to quenched disordered directed percolation but merely to have the same universality class. The percolation thresholds may be different. (Thus our similarity to percolation is different from that of [7] where the damage-spreading thresholds were found to agree with percolation thresholds involving diodes, triodes, tetrodes and pentodes.)

According to [6], the critical exponents of quenched disordered directed percolation differ from the normal directed percolation; for example, the time fractal dimension $\nu_\parallel/\nu_\perp$ was found to be $2+\varepsilon/2$ instead of $2+\varepsilon/4$, and also the mass fractal dimension $d-\beta/\nu_\perp = 2-\varepsilon/2$ is different; $\varepsilon = 4-d$ in both cases.

We do not know of numerical studies of quenched disordered directed percolation to compare with the known [8] fractal dimensions of the Kauffman models, except the simulations of Noest for two dimensions [5] which gave different exponents. However, $d = 4$ should be the upper critical dimension [6] such that for larger d the exponents stay at their values for $d = 4$. For $d = 4$ one therefore expects logarithmic correction factors. In Hansen's numerical study [9] of the four-dimensional Kauffman model the lattice size could not be varied by orders of magnitude; thus his Monte Carlo exponents (1.8 for mass and 2.1 for time fractal dimension) can be effective values influenced by the logarithmic corrections: if a quantity A varies as $L^D(\log L)^x$ with length L, then the effective fractal dimension is $d_f = d(\log A)/d(\log L) = D + x/(\log L)$.

Instead of this possibility, the present problem allows also an alternative explanation: in four dimensions the logarithmic corrections to the initial coupling constants given in the renormalisation group equations are extremely small because of the numerical factor $K_4 = S_4/(2\pi)^4 = 1/32\pi^2 = 1/316$ which enters into the perturbation equation of the coupling constant $u^R = u[1-u\ln(\ldots)+\ldots]$ with $G_1 = g_0^2 K_4$, $G_2 = s_0^2 K_4$, where u stands for G_1 and G_2. In numerical simulations a logarithm will hardly be larger than $1/G$ or 316. However, it causes approximately stable values of the longitudinal and perpendicular Fisher exponents $\eta_\parallel$ and $\eta_\perp$ of [6] in a very large intermediate region:

$$\eta_\parallel = -G_1/8 + G_2$$
$$\eta_\perp = -G_1/16.$$

The unrenormalised value of g_0 is of order unity, $s_0 \simeq N^{-1/2}$, and they are practically unchanged *from scale to scale*. This effect causes effective exponents

$$\nu_\parallel = 1 + \eta_\parallel/2$$
$$\nu_\perp = (1+\eta_\perp/2)/2$$
$$\beta = 1 + \eta_\parallel/4 + \eta_\perp/2$$

very slightly different from mean-field theory; however, $\gamma = 1$. For mass and time fractal dimensions we get $2-G_1/16+G_2$ and $d-\beta/\nu_\perp$, respectively. This intermediate regime can from any practical point of view be regarded as asymptotic. It is, of course, non-universal with effective exponents depending on the lattice and disorder. In principle, the interesting possibility exists that disorder can be made large enough (or branching constant small enough) to change the sign of η and the sign of the corrections to the mean-field exponents. Indeed the numerical values 1.8 and 2.1 differ from the theoretical expectation 2 in the same direction as the ε expansion predicts.

It must be noted that in deterministic systems obeying randomly fixed rules there can appear effects which cannot exist at all in probabilistic models. For example, we can have, with a probability decaying exponentially with N', two sites which are influenced only by each other and are themselves independent of the behaviour of the other neighbours, though they may influence them. If the initial perturbation changes the behaviour of these two points it will lead to permanent damage in the system. Then, even for $p < p_c$, percolation in the sense of a cluster surviving for infinite times is now guaranteed but no spatial spreading of the damage occurs. From the point of view of percolation theory this perturbation has another dimensionality (linear instead of a point perturbation) and this can produce different values of the damage exponents. This explains the absence of a change in the critical behaviour found in [10] for point and linear perturbation in two dimensions; similar three-dimensional [9, 11] simulations would be helpful. Also the effects of non-universality at $d = 4$ for both the Kauffman and quenched disordered percolation should be checked. So one must be very cautious in the application of our analogy.

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